

Quantum Range-Doppler Algorithm

An End-to-End Quantum Circuit for SAR Image Formation

Mohamad El Moussawi¹, Ali J. GHANDOUR², Khalil Al Salahat¹

The 6th International Electronic Conference on Remote Sensing, 19 October 2026

1. Geoinformatics Engineering Students at Politecnico di Milano, Italy
2. National Center for Remote Sensing, Lebanon

SAR image formation

- SAR: imaging in all weather, day and night
- Raw echoes $s(\tau, \eta)$ must be **focused**
- **Range Doppler Algorithm**: FFT $\rightarrow$ filter $\rightarrow$ IFFT
- FFT cost $O(N \log N)$: a bottleneck for large scenes

The quantum promise

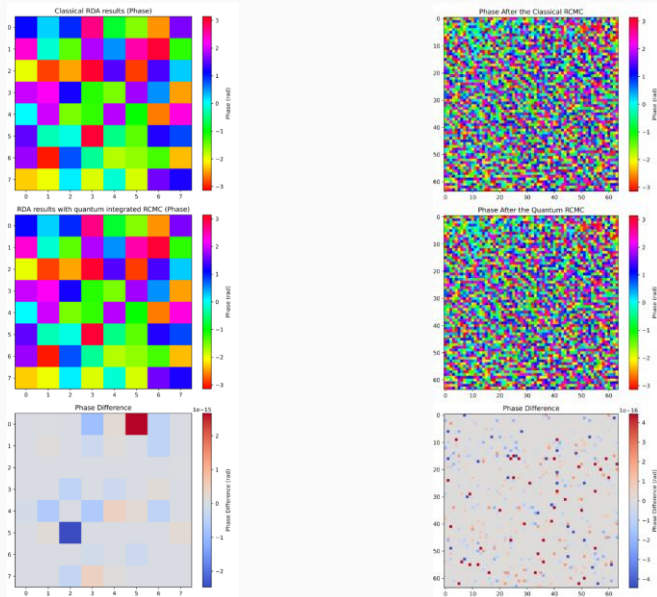
The QFT acts on $N = 2^n$ amplitudes with $O(n^2)$ gates.

But only if the data stay coherent from encoding to readout.

From our previous work to this paper

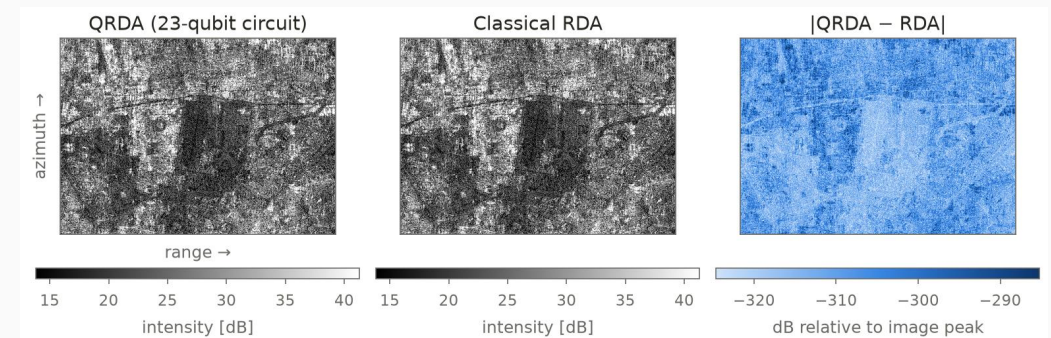
Previous work: Quantum RCMC

- RCMC as a diagonal unitary gate
- Validated on Sentinel-1: 64×64 in isolation, 8×8 inside a classical RDA
- **Rest of the RDA stayed classical**: data read out and re-loaded



This paper: the full QRDA

- **All** stages in one circuit
- **No intermediate measurement**
- Exact equivalence with the classical RDA, proven and verified



The classical chain we reproduce

$$\begin{aligned} S_1 &= \Pi_\eta \mathcal{F}_\eta \mathcal{F}_\tau[s] && \text{2D FFT} \\ S_2 &= S_1 \odot G && \text{range compression} \\ S_3 &= S_2 \odot G_{\text{RCMC}} && \text{RCMC} \\ s_4 &= \Pi_\tau \mathcal{F}_\tau^{-1}[S_3] && \text{back to range} \\ s_5 &= s_4 \odot H_{\text{az}} && \text{azimuth compression} \\ s_{\text{ac}} &= \mathcal{F}_\eta^{-1} \Pi_\eta[s_5] && \text{focused image} \end{aligned}$$

- Filters derived from Sentinel-1 Level-0 metadata and ephemeris: $R_0, V_r, D(f_\eta, V_r)$
- $G, G_{\text{RCMC}}, H_{\text{az}}$ are all **phase-only**, $|\cdot| = 1$

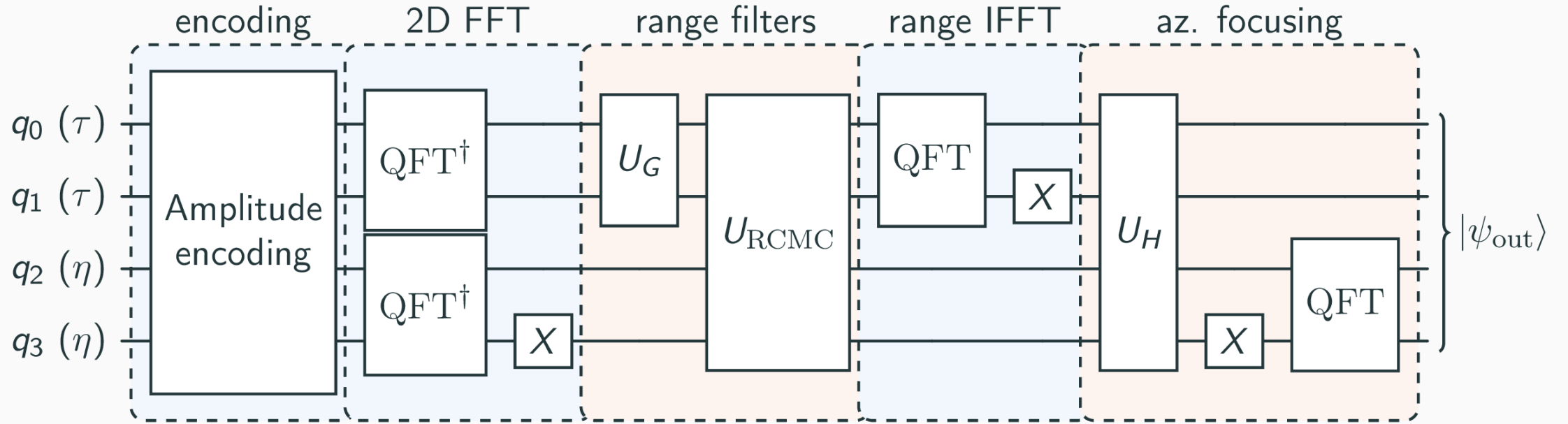
Idea

Phase-only filter = diagonal unitary gate

FFT = QFT

The QRDA circuit

The full QRDA: one circuit, one operator

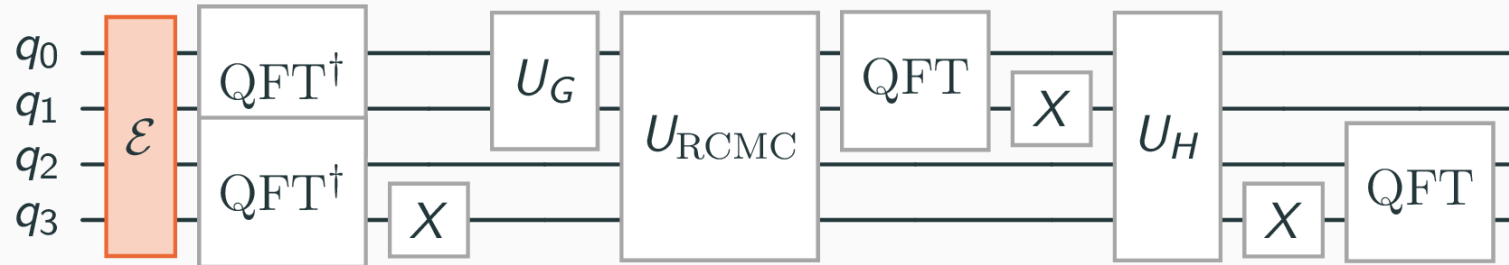


$$U_{\text{QRDA}} = \underbrace{(\text{QFT}_{n_a} \Pi \otimes I)}_{\textcircled{7} \text{ az. IFFT}} \underbrace{U_H}_{\textcircled{6} \text{ az. comp.}} \underbrace{(I \otimes \Pi \text{QFT}_{n_r})}_{\textcircled{5} \text{ back to range}} \underbrace{U_{\text{RCMC}}}_{\textcircled{4} \text{ RCMC}} \underbrace{U_G}_{\textcircled{3} \text{ range comp.}} \underbrace{(\Pi \text{QFT}_{n_a}^\dagger \otimes \text{QFT}_{n_r}^\dagger)}_{\textcircled{1} \text{ 2D FFT} + \textcircled{2} \text{ shift}}$$

$$|\psi_{\text{out}}\rangle = U_{\text{QRDA}} |\psi_0\rangle,$$

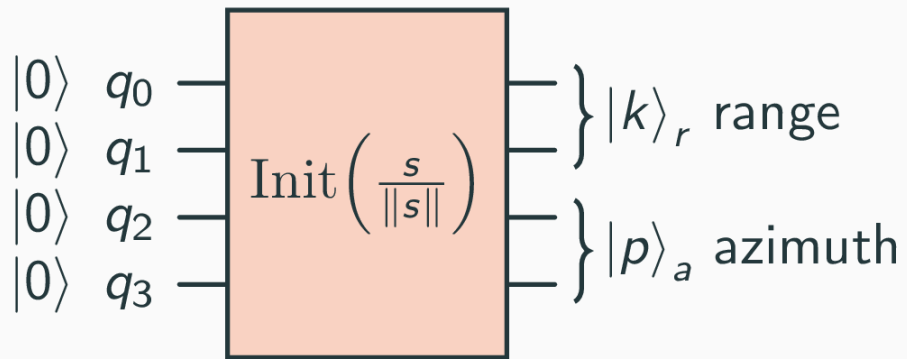
$$s_{\text{ac}}[p, k] = \|s\| \langle p, k | \psi_{\text{out}} \rangle$$

① Amplitude encoding: loading the raw echoes

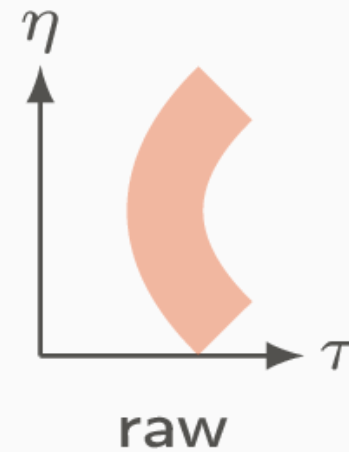


Focusing effect: none yet

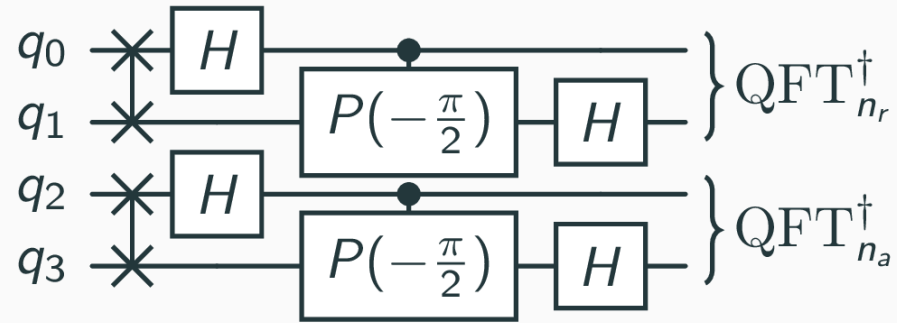
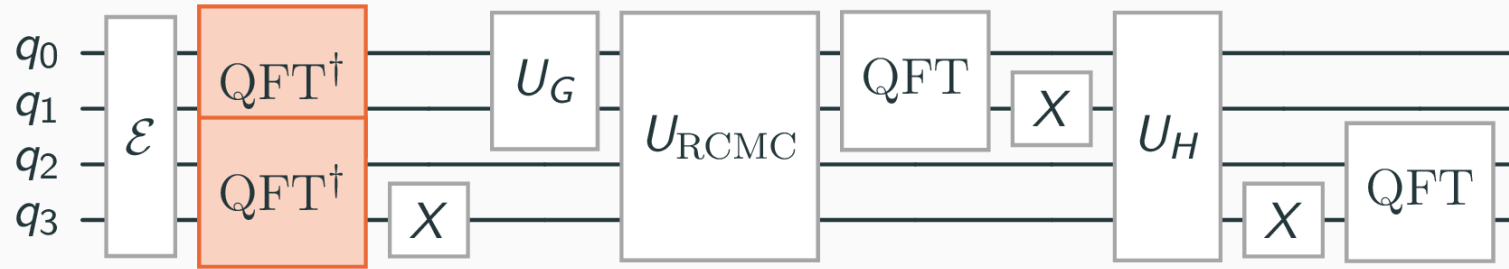
- Each target's energy is spread over many amplitudes: chirp in range, synthetic aperture in azimuth



$$|\psi_0\rangle = \frac{1}{\|s\|} \sum_{p=0}^{N_a-1} \sum_{k=0}^{N_r-1} s[p, k] |p\rangle_a |k\rangle_r$$



① 2D FFT: inverse QFTs on both registers

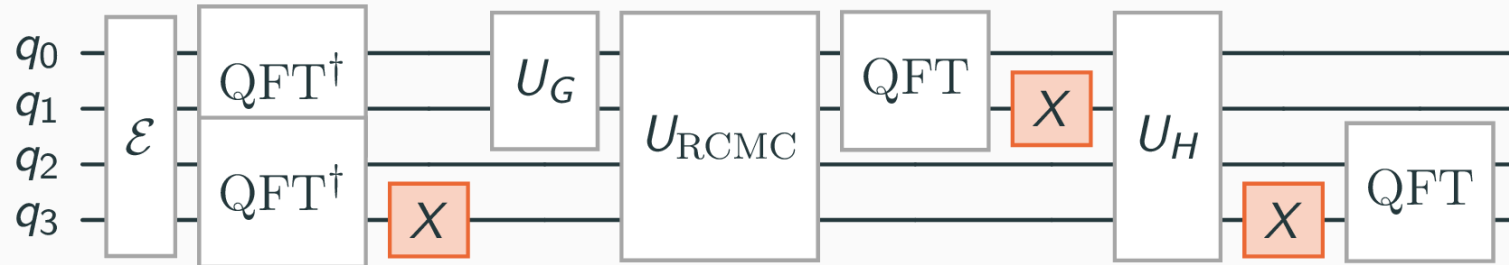


$$\text{QFT} |x\rangle = \frac{1}{\sqrt{M}} \sum_k e^{+2i\pi xk/M} |k\rangle$$

$$\mathcal{F}_M = \sqrt{M} \text{QFT}^\dagger$$

$$|\psi'_1\rangle = \left(\text{QFT}^\dagger_{n_a} \otimes \text{QFT}^\dagger_{n_r} \right) |\psi_0\rangle = \frac{\mathcal{F}_\eta \mathcal{F}_\tau[s]}{\sqrt{N} \|s\|}$$

② Frequency shift: one X gate



$$\left. \begin{array}{l} q_2 \text{ (LSB)} \\ q_3 \text{ (MSB)} \end{array} \right\} \Pi_\eta$$

$$\Pi |k\rangle = |(k + M/2) \bmod M\rangle$$

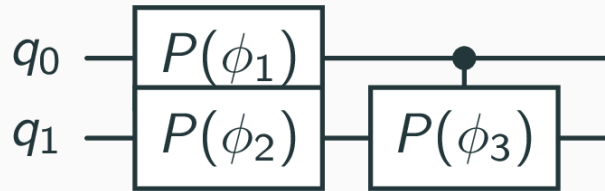
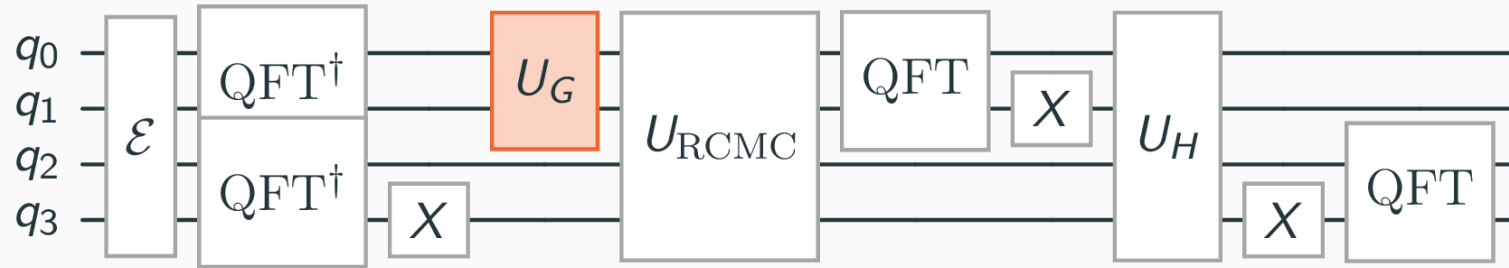
Adding $M/2 = 2^{m-1}$ flips only the **most significant bit**:

$$\Pi = X_{\text{MSB}}, \quad |\psi_1\rangle = (\Pi \otimes I) |\psi'_1\rangle = \frac{S_1}{\sqrt{N}\|s\|}$$

Focusing effect: aligns the filters

- Moves zero Doppler to the centre
- G_{RCMC} and H_{az} depend on $D(f_\eta, V_r)$, defined on this centred grid
- Without it, each Doppler bin would receive the correction of another frequency

③ Range compression gate U_G



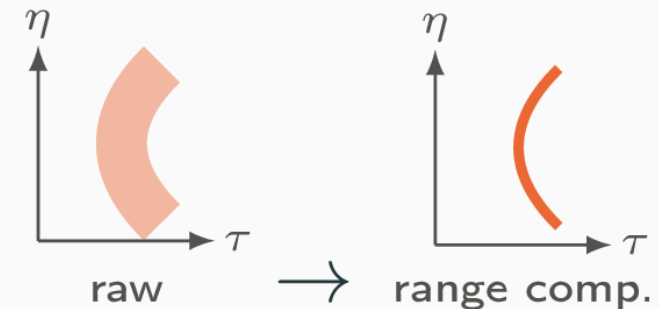
$$G_k = e^{i\theta_k}: \phi_1 = \theta_1 - \theta_0, \phi_2 = \theta_2 - \theta_0, \\ \phi_3 = \theta_3 - \theta_2 - \theta_1 + \theta_0, \text{ global phase } e^{i\theta_0}$$

$$U_G = I_{N_a} \otimes \text{diag}(G_0, \dots, G_{N_r-1})$$

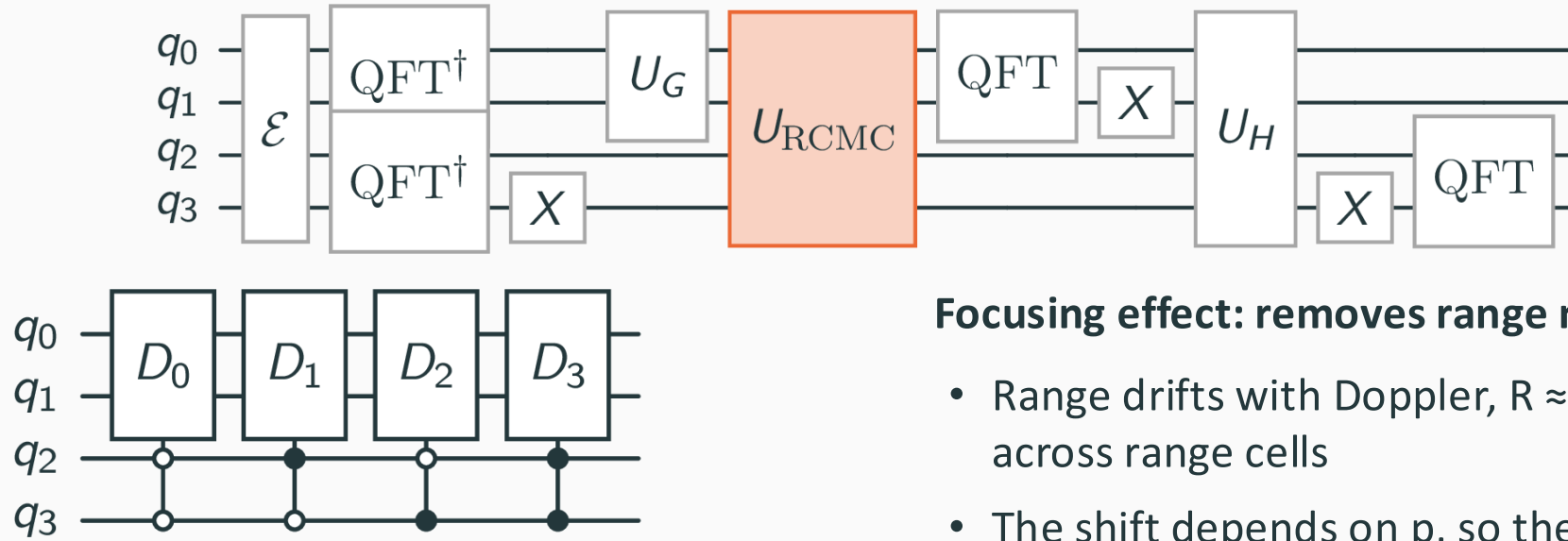
$$G_k = \frac{\overline{\mathcal{F}\{s_{\text{ref}}\}_k}}{|\mathcal{F}\{s_{\text{ref}}\}_k|}, \quad U_G |p\rangle |k\rangle = G_k |p\rangle |k\rangle$$

Focusing effect: range

- The echo is a chirp $e^{i\pi K t^2}$ of length T_p : its spectrum has a **quadratic phase**
- G_k is the conjugate phase: only the **linear phase** of the target delay remains
- After the range IFFT ⑤: a sharp peak instead of a chirp



④ Quantum RCMC gate U_{RCMC}



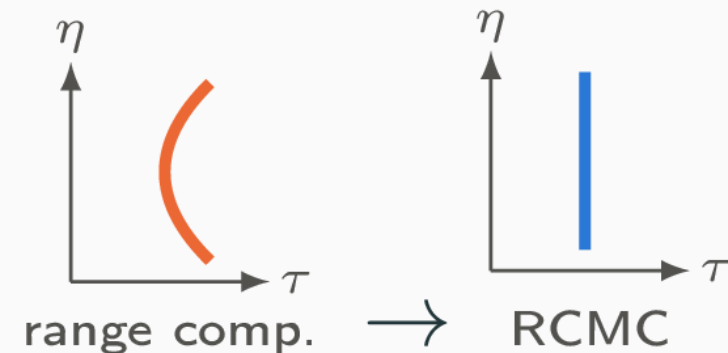
Focusing effect: removes range migration

- Range drifts with Doppler, $R \approx R_0/D(f_\eta)$: a curve across range cells
- The shift depends on p , so the gate is **controlled by the azimuth qubits**

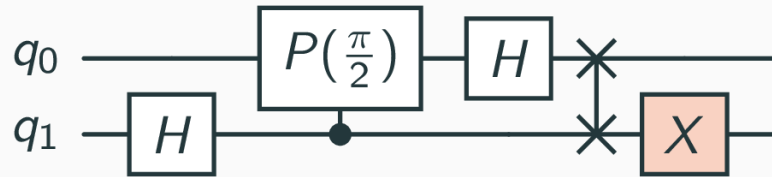
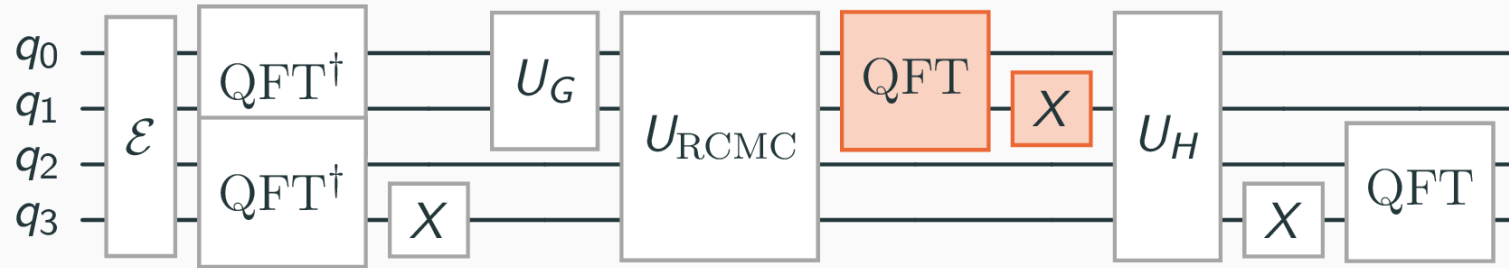
$$U_{\text{RCMC}} = \sum_p |p\rangle \langle p|_a \otimes \underbrace{\text{diag} (e^{i\theta_{p,0}}, \dots, e^{i\theta_{p,N_r-1}})}_{D_p}$$

$$\theta_{p,k} = \frac{4\pi f_{\tau,k}}{c} \tilde{R}_0 \left(\frac{1}{D_{p,k}} - 1 \right)$$

$f_{\tau,k}$: FFT-bin frequency of range bin k (order $0, \dots, \frac{f_s}{2}, -\frac{f_s}{2}, \dots$), exact for fractional shifts; $\tilde{R}_0$: focused slant range of the first range cell



⑤ Back to the range time domain

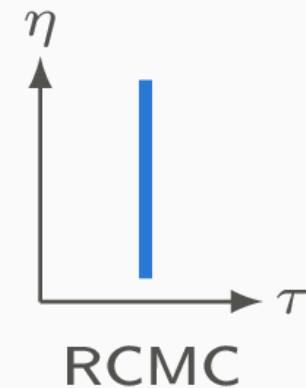


$$\mathcal{F}_M^{-1} = \frac{1}{\sqrt{M}} \text{QFT} \Rightarrow I \otimes \Pi \text{QFT}_{n_r} = \sqrt{N_r} \Pi_\tau \mathcal{F}_\tau^{-1}$$

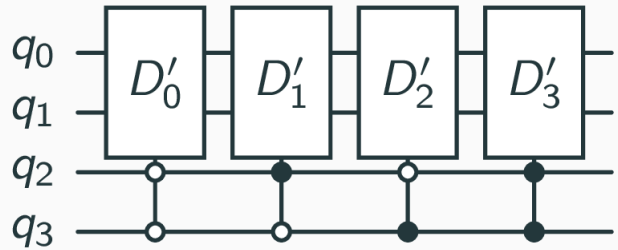
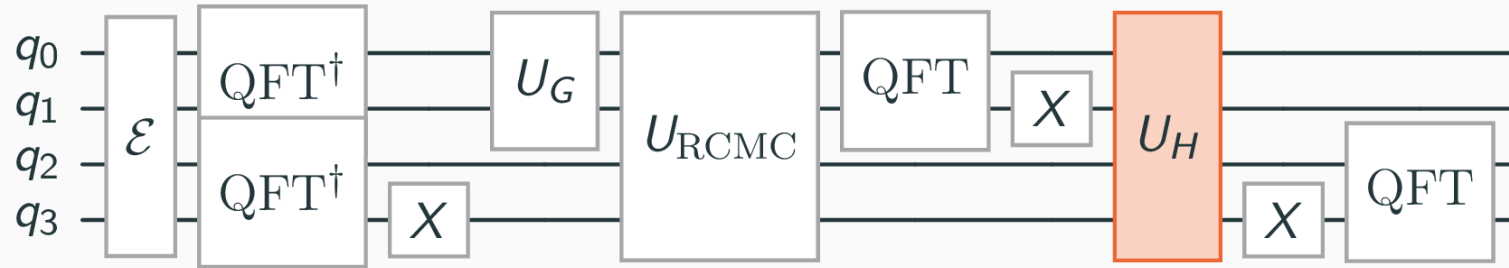
$$|\psi_4\rangle = (I \otimes \Pi \text{QFT}_{n_r}) |\psi_3\rangle = \frac{s_4}{\sqrt{N_a} \|s\|}$$

Focusing effect: range focusing completed

- IFFT $\rightarrow$ QFT (no dagger), then $\Pi_\tau = X$ on q_1
- The products of ③ ④ appear in range time: targets **compressed** and **straightened**
- Data now in the **range-Doppler domain**



⑥ Azimuth compression gate U_H



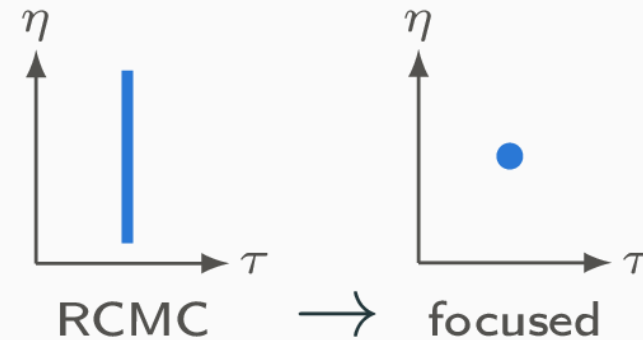
$$U_H = \sum_p |p\rangle\langle p|_a \otimes \underbrace{\text{diag}(H_{\text{az}}[p, 0], \dots, H_{\text{az}}[p, N_r - 1])}_{D'_p}$$

$$H_{\text{az}}[p, k] = \exp \left[4i\pi \frac{\tilde{R}_k D_{p,k}}{\lambda} \right]$$

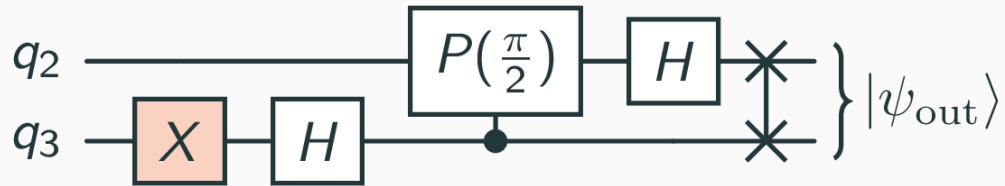
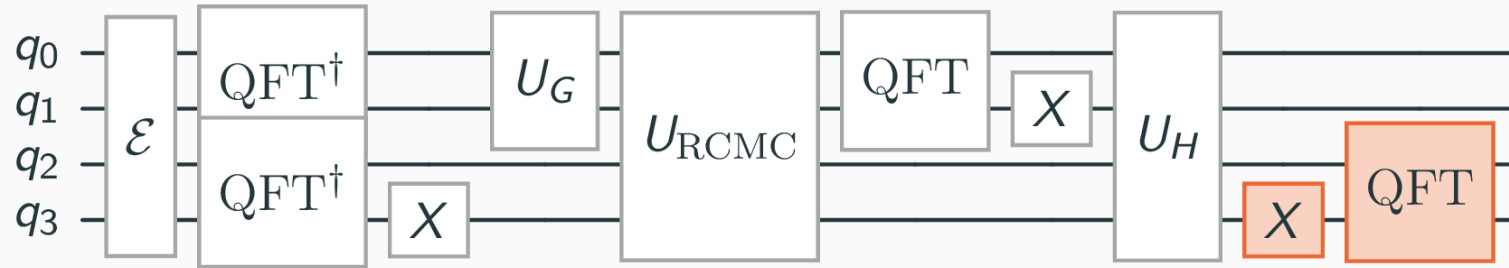
$\tilde{R}_k = R(\tau_k) - \frac{N_{\text{tx}}}{2} \frac{c}{2f_s}$: range of the target in column k after range compression

Focusing effect: azimuth

- A target at R_0 carries the azimuth phase history $e^{-4i\pi R_0 D(f\eta)/\lambda}$
- H_{az} cancels it: only the linear phase of the azimuth position remains
- Depends on $\tilde{R}_k$ **and** $f_{\eta,p}$: acts on all qubits



⑦ Azimuth IFFT and readout



$$\text{QFT}_{n_a} \Pi \otimes I = \sqrt{N_a} \mathcal{F}_\eta^{-1} \Pi_\eta \Rightarrow |\psi_{\text{out}}\rangle = \frac{s_{\text{ac}}}{\|s\|}$$

Proposition (exactness)

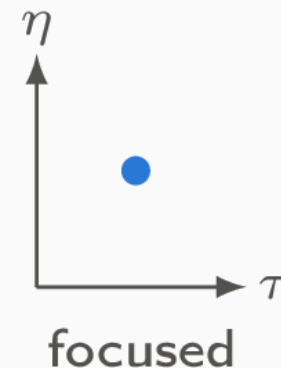
$$s_{\text{ac}}[p, k] = \|s\| \langle p, k | \psi_{\text{out}} \rangle$$

scale factors: $\frac{1}{\sqrt{N_a N_r}} \cdot \sqrt{N_r} \cdot \sqrt{N_a} = 1$

Readout: statevector (simulation), repeated measurements (hardware)

Focusing effect: the image

- Π_η (X on q_3) undoes the shift of ②
- The linear phase left by U_H becomes a **peak at the target position**
- QFT scale factors cancel, unitary after encoding: $\|s_{\text{ac}}\| = \|s\|$



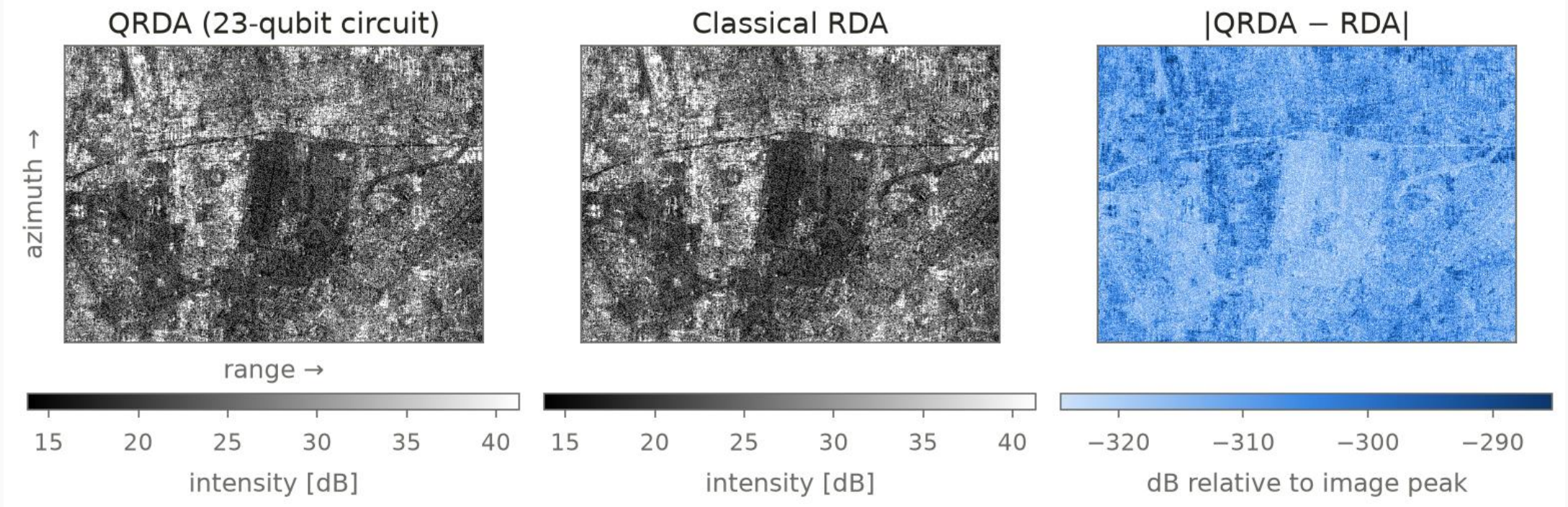
Results

Validation: QRDA = classical RDA to machine precision

Sentinel-1A stripmap S2, HH, 18 March 2015. Filters from the Level-0 metadata and orbit.

Subset $N_a \times N_r$	Qubits	Relative error	Max. phase diff.	$1 - F$	Time
4×4	4	5.0×10^{-16}	7.9×10^{-16} rad	$< 10^{-15}$	0.05 s
16×16	8	1.0×10^{-15}	3.9×10^{-15} rad	$< 10^{-15}$	0.03 s
64×64	12	2.8×10^{-15}	6.2×10^{-14} rad	$< 10^{-15}$	0.04 s
256×512	17	2.9×10^{-15}	6.2×10^{-13} rad	$< 10^{-15}$	0.3 s
1024 × 8192	23	3.8×10^{-14}	3.1×10^{-12} rad	$< 10^{-15}$	17 s

Real data: a Sentinel-1 stripmap image from a 23-qubit circuit



- 1024×8192 subset: 8.4 million amplitudes on 23 qubits
- Range compression with the full chirp
- Same image as the classical RDA
- Difference ≤ -270 dB from the peak (relative error 3.8×10^{-14}): floating-point rounding

Limitations and the road ahead

Limitations

- **Readout:** intensities need many shots, phases need interferometric readout
- Noise-free simulation only; memory $\propto 2^n$, the number of pixels

Directions

- **IW / TOPS mode:** antenna-steering correction as extra gates
- Noise studies on realistic hardware models

Conclusions

- A **complete** Quantum Range-Doppler Algorithm: every RDA stage in one circuit, no intermediate measurement
- $\log_2 N$ qubits
- Equivalent to the classical RDA to machine precision, real Sentinel-1 stripmap data up to **23 qubits**

Thank you

Questions?